THE MODULI OF CURVES OF GENUS 6 AND K3 SURFACES

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ABSTRACT. We prove that the coarse moduli space of curves of genus 6 is birational to an arithmetic quotient of a bounded symmetric domain of type IV by giving a period map to the moduli space of some lattice-polarized K3 surfaces.

INTRODUCTION

This paper gives a birational period map between the coarse moduli space of curves of genus six and the moduli space of some lattice-polarized K3 surfaces. This kind of correspondence was given by the second author for curves of genus 3 and genus 4 in [Ko1] and [Ko2]. A part of the results in this paper was announced in [Ko2].

Let C be a general curve of genus six, then its canonical model is a quadratic section of a unique quintic Del Pezzo surface $Y \subset \mathbb{P}^5$ (e.g. [SB]). The double cover of Y branched along C is a K3 surface X. By taking the period point of X we define a period map $\mathcal{P}$ from an open dense subset of the coarse moduli space $\mathcal{M}_6$ of curves of genus six to an arithmetic quotient of a bounded symmetric domain $\mathcal{D}$ of type IV

$$\mathcal{P} : \mathcal{M}_6 \dashrightarrow \mathcal{D}/\Gamma.$$

The same construction defines rational period maps

$$\mathcal{P}^* : \mathcal{W}_6^2 \dashrightarrow \mathcal{D}/\Gamma^*, \quad \mathcal{P}^{**} : \widetilde{\mathcal{M}}_6 \dashrightarrow \mathcal{D}/\Gamma^{**}.$$

Here the moduli space $\mathcal{W}_6^2$ parametrizes pairs (C, D) where C is a curve of genus six and D is a g_2^6 on C, while $\widetilde{\mathcal{M}}_6$ is the moduli space of plane sextics with four ordered nodes. The group Γ^* is a subgroup of Γ of index 5 and Γ^{**} is a normal subgroup of Γ with $\Gamma/\Gamma^{**} \cong S_5$. In this paper we prove that $\mathcal{P}, \mathcal{P}^*, \mathcal{P}^{**}$ are birational maps and we study their behaviour both generically and at the boundary.

In the first section we review some classical properties of curves of genus six, in particular we recall the structure of the space $\mathcal{W}_6^2$. The natural projection map $\mathcal{W}_6^2 \to \mathcal{M}_6$ is surjective by Brill-Noether theory. Its fiber over the general curve C of genus six is a finite set of cardinality 5 and any of its points gives a birational map from C to a plane sextic with 4 nodes. The fiber is known to be positive dimensional if and only if the curve of genus six is *special*, i.e. it is either trigonal, hyperelliptic, bi-elliptic or isomorphic to a plane quintic curve.

In section 2 we define the period maps $\mathcal{P}, \mathcal{P}^*, \mathcal{P}^{**}$. In fact, we show that the map $\mathcal{P}^{**}$ is equivariant with respect to the natural actions of S_5 and the maps $\mathcal{P}, \mathcal{P}^*$ are obtained by taking the quotient for

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the action of subgroups. Afterwards, we prove that $\mathcal{P}, \mathcal{P}^*, \mathcal{P}^{**}$ are birational maps, in particular $\mathcal{P}$ induces an isomorphism

$$\mathcal{M}_6 \setminus \{\text{special curves}\} \cong (\mathcal{D} \setminus \mathcal{H})/\Gamma$$

where $\mathcal{H}$ is a divisor defined by hyperplane sections associated to (-2)-vectors, called *discriminant divisor*.

In section 3 we study the discriminant divisor $\mathcal{H}$ and its geometric meaning. We prove that $\mathcal{H}/\Gamma$ has 3 irreducible components which parametrize respectively curves of genus six with a node, pairs (C, L) where C is a plane quintic and L is a line and pairs (C, D) where C is a trigonal curve of genus six and $D \in |K_C - 2g^1_3|$.

In the final section we determine the structure of the boundary of the Satake-Baily-Borel compactification of D/Γ and we compare this compactification with the GIT compactification of the space of plane sextics.

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Notation. A lattice L is a free abelian group of finite rank equipped with a non degenerate bilinear form, which will be denoted by $(\, , \,)$.
- The discriminant group of L is the finite abelian group $A_L = L^*/L$, where $L^* = \mathrm{Hom}(L, \mathbb{Z})$, equipped with the quadratic form $q_L : A_L \to \mathbb{Q}/2\mathbb{Z}$ defined by $q_L(x + L) = (x, x) \mod 2\mathbb{Z}$.
- $\mathrm{O}(L)$ and $\mathrm{O}(q_L)$ will denote the groups of isometries of L and A_L respectively.
- A lattice is unimodular if $|A_L| = |\det L| = 1$.
- If M is the orthogonal complement of L in a unimodular lattice, then $A_L \cong A_M$ and $q_M = -q_L$.
- We will denote by U the hyperbolic plane and by A_n, D_n, E_n the negative definite lattices of rank n associated to the Dynkin's diagrams of the corresponding types.
- The lattice $L(\alpha)$ is obtained multiplying by α the form on L.
- The lattice L^m is the orthogonal direct sum of m copies of the lattice L.

We will refer the reader to [N1] for basic facts about lattices.

1. Curves of genus six and quintic Del Pezzo surfaces

We start recalling some well-known properties of curves of genus six. By Brill-Noether theory any smooth curve of genus six C has a special divisor D with $\deg(D) = 6$ and $h^0(C, D) = 3$. Let φ_D be the morphism asociated to D:

$$\varphi_D : C \longrightarrow \mathbb{P}^2.$$

The curve C will be called *special* if it is either hyperelliptic, trigonal, bi-elliptic or isomorphic to a smooth plane quintic curve. The following is given for example in section A, Ch.V in [ACGH].

Proposition 1. *Let C be a smooth curve of genus six, then one of the followings holds*:

$a)$ *φ_D is birational and $\varphi_D(C)$ is an irreducible plane sextic having only double points.*
$b)$ *C is special.*

Case a) Assume first that $\varphi_D(C)$ is a plane sextic with 4 nodes $p_1, \ldots, p_4$ in general position. The blowing up of $\mathbb{P}^2$ in these points is a quintic del Pezzo surface Y and $C \in |-2K_Y|$. In fact, the embedding $C \subset Y \subset \mathbb{P}^5$ is the canonical embedding of C and Y is the unique quintic Del Pezzo

surface containing C (see e.g. [SB]). Let e_0 be the class of the pull back of a line and let e_i be the classes of exceptional divisors over the points p_i. The surface Y contains 10 lines

$$e_i\ ,\ e_0 - e_i - e_j, \quad 1 \le i < j \le 4.$$

It is known that the group of automorphisms of the dual graph of the 10 lines is isomorphic to S_5. The surface Y admits exactly five birational morphisms to $\mathbb{P}^2$, called *blowing down maps*, induced by the linear systems:

$$e_0,\ 2e_0 - \sum_{i=1}^{4} e_i + e_j, \quad j = 1, \ldots, 4.$$

Note that any such morphism maps C to a plane sextic with 4 nodes. In fact also the converse holds i.e. there is a one-to-one correspondence between the set of blowing down maps for Y and the set $W^2_6(C)$ of g^2_6 on C. In particular the generic curve of genus 6 has exactly five g^2_6. The automorphisms group of Y acts on the blowing down classes, giving a representation $Aut(Y) \to S_5$, which is known to be an isomorphism. The stabilizer of a blowing down model ϕ is given by projectivities permuting the 4 points $p_1, \ldots, p_4 \in \mathbb{P}^2$ which are the image of the exceptional divisors of ϕ, while an element of order five is realized by a quadratic transformation α with fundamental points at p_1, p_2, p_3 [Do, Theorem 10.2.2].

If $p_1, \ldots, p_4$ are not in general position then either 3 of them lie on a line or two of them are infinitely near. Note that anything worse is not admitted since $\varphi_D(C)$ is irreducible with at most double points. The blowing up of $\mathbb{P}^2$ in these points is a *nodal* del Pezzo surface, i.e. $-K_Y$ is nef and big (see [DO]). Equivalently, the anti-canonical model of Y has at most rational double points. In this case the properties of the embedding $C \subset Y$ still hold, in particular Y containing C is unique ([AH, 5.14]). However, the surface Y may have less than five blowing down classes, i.e. C has less than five g^6_2.

Case b) The following characterization holds:

Proposition 2. *A curve of genus six C is special if and only if* $\dim W^2_6(C) > 0$.

Proof. We have seen that if C is not special, then $\dim W^2_6(C) = 0$ and contains at most five points. We now see what happens for special curves ([ACGH]).

- If C is *trigonal* then it has two types of g^2_6:

$$D = 2g^1_3 \quad \text{and} \quad D(p) = K_C - g^1_3 - p,\ \ p \in C.$$

  Hence $W^2_6(C)$ is one dimensional and has two irreducible components. The plane model $\varphi_D(C)$ is a triple conic and $\varphi_{D(p)}(C)$ is a plane sextic with a triple point and a node.
- If C is isomorphic to a *plane quintic* then any g^2_6 on C is of type: $D(p) = g^2_5 + p$, $p \in C$. Hence $W^2_6(C) \cong C$. The plane model $\varphi_{D(p)}(C)$ is a plane quintic.
- If C is *bi-elliptic* i.e. there exists $\pi : C \to E$, where E is an elliptic curve, then any g^2_6 corresponds to $\phi \circ \pi$ where ϕ is a g^1_2 on E. The plane model of C is a double cubic.
- If C is *hyperelliptic* then any g^2_6 is of type:

$$D(p, q) = K_C - g^1_2 - p - q,\ \ p, q \in C.$$

  Hence $W^2_6(C) \cong Sym^2(C)$. In fact $D = K_C - 2g^1_2$ is a singular point of $W^2_6(C)$. The plane model $\varphi_D(C)$ is a double rational cubic and $\varphi_{D(p,q)}(C)$ is a double conic.

□

Remark 1. It follows that the moduli space of curves of genus six is birational to the GIT moduli space of plane sextics with 4 nodes up to the action of the group generated by projectivities and by the birational transformation α.

2. $K3$ SURFACES ASSOCIATED TO CURVES OF GENUS 6

2.1. **The geometric construction.** Let $C \subset \mathbb{P}^5$ be the canonical model of a non-special smooth curve of genus six. By the remarks in the previous section, there is a unique nodal Del Pezzo surface Y such that C lies in the anti-canonical model of Y in $\mathbb{P}^5$. Let $Y' \to Y$ be the canonical resolution of rational double points of Y. Since $C \in |-2K_{Y'}|$, there exists a double cover

$$\pi : X \longrightarrow Y'$$

branched along C and X is a $K3$ surface. It is well known that $H^2(X, \mathbb{Z})$, together with the cup product, is an even unimodular lattice of signature $(3, 19)$. The covering involution σ of π acts on this lattice with eigenspaces

$$H^2(X, \mathbb{Z})^{\pm} = \{x \in H^2(X, \mathbb{Z}) : \sigma^*(x) = \pm x\}.$$

Lemma 1. $H^2(X, \mathbb{Z})^+ \cong A_1(-1) \oplus A_1^4,\ H^2(X, \mathbb{Z})^- \cong U \oplus U \oplus E_8 \oplus A_1^5.$

Proof. By definition, the lattices $H^{\pm} = H^2(X, \mathbb{Z})^{\pm}$ are 2-elementary, i.e. their discriminant groups are 2-elementary abelian groups. By [N1, Theorem 3.6.2] the isomorphism class of a 2-elementary even indefinite lattice L is determined uniquely by the triple (s, ℓ, δ), where s is the signature, ℓ is the minimal number of generators of A_L and δ is 0 (resp. 1) if the quadratic form on A_L always assumes integer values (resp. otherwise). On the other hand [N2, Theorem 4.2.2] shows that H^+ has $s = (1, 4),\ \ell = 5,\ \delta = 1$. Since H^- is the orthogonal complement of H^+ in the unimodular lattice $H^2(X, \mathbb{Z})$, it has $s = (2, 15),\ \ell = 5,\ \delta = 1$. Hence it is enough to check that the lattices in the right hand sides have the same triple of invariants. □

Let S_X be the Picard lattice of X and let T_X be its transcendental lattice:

$$S_X = H^2(X, \mathbb{Z}) \cap \omega_X^{\perp}, \qquad T_X = S_X^{\perp}.$$

Note that the invariant lattice $H^2(X, \mathbb{Z})^+$ coincides with the pull-back of the Picard lattice of Y, hence

$$H^2(X, \mathbb{Z})^+ \subset S_X, \qquad T_X \subset H^2(X, \mathbb{Z})^-.$$

If ω_X is a nowhere vanishing holomorphic 2-form on X, then $\omega_X \in T_X \otimes \mathbb{C}$, hence $\sigma^*(\omega_X) = -\omega_X$.

Lemma 2. *There are no* (-2)*-vectors in* $S_X \cap H^2(X, \mathbb{Z})^-$.

Proof. Assume that r is such a vector. By Riemann-Roch theorem we may assume that r is effective. Then $\sigma^*(r) = -r$ is also effective. This is a contradiction. □

2.2. **Lattices.** We will denote by L_{K3} an even unimodular lattice of signature $(3, 19)$. This is known to be unique up to isomorphisms (see e.g. [N1, Theorem 1.1.1]), hence the lattice $H^2(X, \mathbb{Z})$ is isomorphic to L_{K3}. Let $\bar{e}_0, \bar{e}_1, \ldots, \bar{e}_4$ be the pull-backs of the classes $e_0, e_1, \ldots, e_4$ under π^*. These generate a sublattice of S_X isometric to $A_1(-1) \oplus A_1^4$. Let

$$S = A_1(-1) \oplus A_1^4, \quad T = U \oplus U \oplus E_8 \oplus A_1^5.$$

Denote by $s_0, s_1, \dots, s_4$ an orthogonal basis for S with $s_0^2 = 2$, $s_i^2 = -2,\ i = 1, \dots, 4$ and denote by $r_1, \dots, r_5$ an orthogonal basis for the A_1^5 component of T.

Lemma 3. *Let $\xi_i = r_i/2$, then the discriminant group A_T consists of the following vectors*:

$$\begin{array}{ll} q(x) = 0: & 0,\ \sum_{i\neq j}\xi_i, \quad 1 \leq j \leq 5 \\ q(x) = 1: & \xi_i + \xi_j, \quad 1 \leq i < j \leq 5 \\ q(x) = -1/2: & \xi_i, \quad 1 \leq i \leq 5,\ \sum_{i=1}^5 \xi_i \\ q(x) = -3/2: & \sum_{i\neq j,k}\xi_i, \quad 1 \leq j < k \leq 5. \end{array}$$

It follows from [N1, Theorem 1.14.4] that S can be embedded uniquely in L_{K3} and T is isomorphic to its orthogonal complement. Since L_{K3} is unimodular,

$$A_S \cong A_T \cong \mathbb{F}_2^5, \quad q_S \cong -q_T$$

and an isomorphism from A_S to A_T is given by

$$s_0/2 \mapsto \xi_1, \quad (2s_0 - \sum_{i=1}^4 s_i + s_j)/2 \mapsto \xi_{j+1}, \quad j = 1, \dots, 4.$$

Lemma 4. *There are isomorphisms O$(q_S) \cong$ O$(q_T) \cong S_5$ and the natural maps*

$$O(T) \to O(q_T), \quad O(S) \to O(q_S)$$

are surjective.

Proof. The first statement follows from [MS]. Note that $\mathrm{O}(q_T)$ acts on A_T by permuting the ξ_i's. The surjectivity statement for T is obvious, since clearly exist isometries of T permuting the r_i's. On the other hand, the automorphism group S_5 of Y acts on S as isometries. These isometries act on A_S as S_5. More concretely, the isometries of S permuting the s_i's $(1 \leq i \leq 4)$ and the isometry

$$s_0 \mapsto 2s_0 - s_1 - s_2 - s_3,\ s_1 \mapsto s_0 - s_1 - s_3,\ s_2 \mapsto s_4,\ s_3 \mapsto s_0 - s_2 - s_3,\ s_4 \mapsto s_0 - s_1 - s_2$$

generate $\mathrm{O}(q_S)$. □

In the following we will consider three arithmetic groups acting on T:

$$\Gamma = \mathrm{O}(T), \quad \Gamma^* = \{\gamma \in \mathrm{O}(T):\ \gamma(\xi_1) = \xi_1\}, \quad \Gamma^{**} = \{\gamma \in \mathrm{O}(T):\ \gamma|A_T = 1\}.$$

Note that $\Gamma/\Gamma^{**} \cong \mathrm{O}(q_T) \cong S_5$.

Lemma 5. *Let $O_T = \{\gamma \in O(L_{K3}) : \gamma(T) = T\}$. Then the restriction homomorphisms*

$$O_T \to \Gamma, \quad \{\gamma \in O_T : \gamma(s_0) = s_0\} \to \Gamma^* \text{ and } \quad \{\gamma \in O_T : \gamma|S = 1_S\} \to \Gamma^{**}$$

are surjective.

Proof. Let $\gamma \in \Gamma$. By Lemma 4 there exists $\beta \in \mathrm{O}(S)$ such that $\beta = \gamma$ on $A_S \cong A_T$. Then the isometry $\beta \oplus \gamma$ on $S \oplus T$ lifts to an isometry in O_T. If $\gamma \in \Gamma^*$ or Γ^{**} then β can be choosen such that $\beta(s_0) = s_0$ or $\beta = 1_S$, respectively (see the proof of Lemma 4). □

Remark 2. There are two orbits of vectors with $q(x) = -1/2$ under the action of $\mathrm{O}(q_T)$:

$$O_1 = \{\sum_{i=1}^5 \xi_i\}, \qquad O_2 = \{\xi_1, \dots, \xi_5\}.$$

2.3. **Moduli spaces.** Since both S and T are 2-elementary lattices, the isometry $(1_S, -1_T)$ on $S \oplus T$ can be extended to an isometry ι of L_{K3}. Let $\alpha : H^2(X,\mathbb{Z}) \to L_{K3}$ be an isometry satisfying $\alpha(H^2(X,\mathbb{Z})^+) = S$. Then $\iota \circ \alpha = \alpha \circ \sigma^*$. Since $\sigma^*(\omega_X) = -\omega_X$ then the *period*

$$p_X(\alpha) = \alpha_{\mathbb{C}}(\omega_X)$$

belongs to the set

$$\mathcal{D} = \{\omega \in \mathbb{P}(T \otimes \mathbb{C}) : (\omega, \omega) = 0, (\omega, \bar{\omega}) > 0\},$$

called the *period domain of S-polarized K3 surfaces*. By Lemma 2, there are no (-2)-vectors orthogonal to the period, hence $p_X(\alpha)$ belongs to the complement of the divisor

$$\mathcal{H} = \bigcup_{r \in T,\ r^2=-2} \mathcal{H}_r \ \text{ where } \ \mathcal{H}_r = \{\omega \in \mathcal{D} : (r, \omega) = 0\}.$$

Consider the orbit spaces

$$\mathcal{M} = \mathcal{D}/\Gamma, \qquad \mathcal{M}^* = \mathcal{D}/\Gamma^*, \qquad \mathcal{M}^{**} = \mathcal{D}/\Gamma^{**}.$$

Let $\mathcal{W}_6^2$ be the moduli space of pairs (C, D) where C is a smooth curve of genus 6 and $D \in W_6^2(C)$ (see [ACGH]) and let $\widetilde{\mathcal{M}}_6$ be the moduli space of plane sextics with four ordered nodes.

Theorem 1. *The geometric construction in* 2.1 *defines a birational map*

$$\mathcal{P}^{**} : \widetilde{\mathcal{M}}_6 \dashrightarrow \mathcal{M}^{**}.$$

The map $\mathcal{P}^{**}$ is equivariant for the natural action of S_5, taking quotients for this action and for the action of a subgroup isomorphic to S_4 gives birational maps

$$\mathcal{P} : \mathcal{M}_6 \dashrightarrow \mathcal{M}, \qquad \mathcal{P}^* : \mathcal{W}_6^2 \dashrightarrow \mathcal{M}^*.$$

In fact it induces an isomorphism

$$\mathcal{M}_6 \setminus \{\textit{special curves}\} \cong \mathcal{M} \setminus (\mathcal{H}/\Gamma).$$

Proof. Let C be a plane sextic with 4 ordered nodes. The construction in 2.1 associates to C a K3 surface X which is birational to the the double cover of $\mathbb{P}^2$ branched along the plane sextic. If C is general, then $S_X = H^2(X,\mathbb{Z})^+$ is the pull-back of the Picard lattice of Y and $\{\bar{e}_0, \bar{e}_1, \dots, \bar{e}_4\}$ gives an ordered basis of S_X.

In general, by using Lemma 5, choose a marking $\alpha : H^2(X,\mathbb{Z}) \to L_{K3}$ such that $\alpha(H^2(X,\mathbb{Z})^+) \subset S$ and $\alpha(\bar{e}_i) = s_i$, $0 \le i \le 4$. By Lemma 2 $\alpha_{\mathbb{C}}(\omega_X) \in \mathcal{D} \setminus \mathcal{H}$. Moreover, if α_1, α_2 are two markings of this type, then $\alpha_2\alpha_1^{-1}$ preserves the ordered basis $\{s_i\}$, hence its restriction to T belongs to Γ^{**}. Thus we can associate to C a point in $\mathcal{D}/\Gamma^{**}$, i.e. we defined a rational map $\mathcal{P}^{**} : \tilde{\mathcal{M}}_6 \dashrightarrow \mathcal{M}^{**}$.

Conversely, let $\omega \in \mathcal{D} \setminus \mathcal{H}$. By the surjectivity theorem of the period map ([Ku, PP]) there exists a marked K3 surface (X, α) such that $\alpha_{\mathbb{C}}(\omega_X) = \omega$. Then $\iota(\omega) = -\omega$ and there exist no (-2)-vectors in $T \cap \omega^\perp$ since $\omega \notin \mathcal{H}$, hence ι preserves an ample class. It now follows from the Torelli theorem [Na, Theorem 3.10] that ι is induced by an automorphism σ on X.

By [N2, Theorem 4.2.2] the fixed locus of σ is a smooth curve C of genus six. The quotient surface $Y = X/(\sigma)$ is smooth and the image of C belongs to $|-2K_Y|$. Hence $-K_Y$ is nef and big with $K_Y^2 = 5$, i.e. Y is a nodal quintic del Pezzo surface. In fact, the pull back of $\mathrm{Pic}(Y)$ is exactly $\alpha^{-1}(S) \subset S_X$.

If we choose $\omega \in \mathcal{D} \setminus \mathcal{H}$ up to the action of Γ^{**} then, by Lemma 5, we get α up to an isometry in O_T which preserves an ordered basis $\{s_i\}$. Hence this gives a K3 surface X with a class $\alpha^{-1}(s_i) \in$

$\alpha^{-1}(S),\ 0 \le i \le 4$. By blowing down the corresponding four (-1)-curves on Y, we get a plane sextic with four ordered nodes. This proves that $\mathcal{P}^{**}$ is birational.

The quotients $\Gamma/\Gamma^{**} \cong S_5$ and $\Gamma^*/\Gamma^{**} \cong S_4$ act on $\mathcal{M}^{**}$ and by Lemma 5 they lift to isometries of L_{K3} which preserve T and S. By taking the quotient for these actions, we get birational maps $\mathcal{P}$ and $\mathcal{P}^*$. □

3. The discriminant divisor

In the previous section we introduced a divisor $\mathcal{H}$ in $\mathcal{D}$. The image of this divisor in $\mathcal{M}$ or $\mathcal{M}^*$ will be called *discriminant divisor*. We now describe its structure and its geometric meaning.

3.1. **Irreducible components.**

Lemma 6. *Let Δ be the set of vectors $r \in T$ with $r^2 = -2$, then*

- *the group Γ has three orbits in Δ :*

$$\Delta_1 = \{r \in \Delta : r/2 \notin T^*\},\ \ \Delta_2 = \{r \in \Delta : r/2 \in O_1\},\ \ \Delta_3 = \{r \in \Delta : r/2 \in O_2\};$$

- *the group Γ^* has 4 orbits in Δ : Δ_1, Δ_2 and two orbits decomposing Δ_3*

$$\Delta_{3a} = \{r \in \Delta : r/2 = \xi_2\},\ \ \Delta_{3b} = \{r \in \Delta : r/2 = \xi_1\}.$$

Proof. Given a vector $r \in \Delta$ we will classify the embeddings of $\Lambda = \langle r \rangle$ in T up to the action of Γ by applying [N1, Proposition 1.15.1]. We first need to give an isometry α between a subgroup of A_Λ and a subgroup of $A_T \cong \mathbb{F}_2^5$. If H is such a subgroup, then either $H = 0$ or $H = \mathbb{F}_2$. Note that $H = \mathbb{F}_2$ if and only if $r/2 \in T^*$.

In case $H = 0$, since there is a unique a lattice K with $q_K = q_\Lambda \oplus (-q_T)$ and $\mathrm{O}(K) \to \mathrm{O}(q_K)$ is surjective by [N1, Theorem 1.14.2], then by [N1, Proposition 1.15.1] there is a unique embedding of Λ in T such that $\Lambda \oplus \Lambda^\perp = T$.

In case $H = \mathbb{F}_2$ there are two different embeddings of Λ, according to the choice of $\alpha(r/2)$ in O_1 or O_2. This gives the first assertion.

The second assertion can be proved in a similar way, by observing that Γ^* has three orbits on the set of vectors $x \in A_T$ with $q(x) = -1/2$. □

For $r \in \Delta_i$, let $T_i = \{x \in T : (x, r) = 0\}$ and denote by S_i the orthogonal complement of T_i in L_{K3}. Then we have:

Lemma 7.

$$\begin{array}{ll} S_1 \cong A_1(-1) \oplus A_1^5, & T_1 \cong U \oplus U \oplus E_7 \oplus A_1^5, \\ S_2 \cong U(2) \oplus D_4, & T_2 \cong U \oplus U(2) \oplus E_8 \oplus D_4, \\ S_3 \cong U \oplus A_1^4, & T_3 \cong U \oplus U \oplus E_8 \oplus A_1^4. \end{array}$$

Proof. Because of Lemma 6 the isomorphism class of T_i does not depend on the choice of $r \in \Delta_i$. If $r \in \Delta_1$ or Δ_3 then we can assume r to be one generator of E_8 or respectively one generator of A_1 in a decomposition $T = U \oplus U \oplus E_8 \oplus A_1^5$. If $r \in \Delta_2$ we can assume r to be a generator of A_1 in a decomposition $T = U \oplus U(2) \oplus E_8 \oplus D_4 \oplus A_1$. In all these cases the orthogonal complement of r in T can be easily computed. The lattices S_i can be computed by applying [N1, Theorem 3.6.2]. □

Corollary 1. *The divisor $\mathcal{H}/\Gamma$ has 3 irreducible components $\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3$ and $\mathcal{H}/\Gamma^*$ has 4 irreducible components $\mathcal{H}_1^*, \mathcal{H}_2^*, \mathcal{H}_{3a}^*, \mathcal{H}_{3b}^*$ such that*

- $\mathcal{H}_i^* \to \mathcal{H}_i,\ \ i = 1, 2$ *have degree* 5,
- $\mathcal{H}_{3a}^* \to \mathcal{H}_3,\ \ \mathcal{H}_{3b}^* \to \mathcal{H}_3$ *have degree* 4 *and* 1 *respectively.*

Let ι_i be the isometry of L_{K3} defined by $\iota_i|S_i = 1_{S_i}$ and $\iota_i|T_i = -1_{T_i}$. The following can be proved by means of Torelli theorem, as in the proof of Theorem 1.

Lemma 8. *There exists a K3 surface X_i such that $S_{X_i} \cong S_i$ and carrying an involution σ_i of X_i with $\sigma_i^* = \iota_i$.*

3.2. **Curves of genus six with a node.** Let C_1 be a generic plane sextic with five nodes. The blowing up of the projective plane at the nodes is a quartic del Pezzo surface Y_1 and its double cover branched along the strict transform of C_1 is a K3 surface X. Alternatively, if we blow up the plane at four nodes, we get a quintic del Pezzo surface on which the strict transform of C_1 is a curve of genus six with a node. The pull-back of Pic(Y_1) is a sublattice of the Picard lattice of X isomorphic to S_1. We now show that also the converse is true

Proposition 3. *The K3 surface X_1 is birational to the double cover of a quintic del Pezzo surface branched along a generic curve of genus six with a node or, equivalently, to a double plane branched along a generic sextic with 5 nodes.*

Proof. Consider the involution σ_1 on X_1 as in Lemma 8. By [N2, Theorem 4.2.2] the fixed locus of σ_1 is a smooth curve C_1 of genus 5. The quotient of X_1 by σ_1 is a smooth rational surface Y_1 and the image of C_1 belongs to $|-2K_{Y_1}|$, hence Y_1 is a del Pezzo surface of degree 4.

Any (-1)-curve e on Y_1 intersects the image of C_1 at two points since $(-K_{Y_1}, e) = 1$. Hence, contracting one (-1)-curve of Y_1 we get a quintic del Pezzo surface where the image of C_1 is a curve of genus six with a node, and contracting five disjoint (-1)-curves C_1 is mapped to a plane sextic with five nodes. □

Corollary 2. *The divisor $\mathcal{H}_1$ is birational to the moduli space of curves of genus six with one node and $\mathcal{H}_1^*$ to the moduli space of plane sextics with 5 nodes, with one marked.*

Proof. Taking the quotient of $\mathcal{H}_1$ for the action of Γ, we identify two markings on X_1 which give the same embedding of $\alpha^{-1}(S)$ in Pic(X_1). This data identifies a (-1)-curve on Y_1, whose contraction gives a quintic Del Pezzo surface and a curve of genus 6 with a node. The group Γ^*, instead, identifies two markings on X_1 if they also give the same embedding of $\alpha^{-1}(h)$ in the Picard lattice. This class gives a blowing down map on Y_1 with a distinguished exceptional divisor.

Using these remarks and Proposition 3, the result follows as in the proof of Theorem 1. □

3.3. **Plane quintics.** Let C_2 be a smooth plane quintic and let L be a line transversal to C_2. The minimal resolution of the double plane branched along $C_2 \cup L$ is a K3 surface X. The Picard lattice of X contains five disjoint (-2)-curves, coming from the resolution of singularities, and a (-2)-curve which is the proper transform of L. These rational curves generate a lattice which is isomorphic to S_2.

Proposition 4. *The surface X_2 is birational to a double plane branched along the union of a plane quintic and a line.*

Proof. This was proved in [L, Ch.6]. □

Corollary 3. *The divisor $\mathcal{H}_2$ is birational to the moduli space of pairs (C, L) where C is a plane quintic and L is a line, while $\mathcal{H}_2^*$ parametrizes triples (C, L, p) where $p \in C \cap L$.*

Proof. The first statement is [L, Corollary 6.21]. The second statement can be proved similarly to Corollary 2. □

3.4. **Trigonal curves of genus six.** Let $C \subset \mathbb{P}^5$ be the canonical model of a trigonal curve of genus 6. Any 3 points in the g_1^3 lie on a line by Riemann-Roch theorem and the closure of the union of all these lines is a quadric Q such that the curve C belongs to $|4f+3e|$, where e, f are the rulings of Q. The minimal resolution of the double cover of Q branched along the union of C with a line $L \in |e|$ is a K3 surface X. The ruling f, the proper transform of L and the exceptional divisors over the four points in $C \cap L$ generate a sublattice of S_X isomorphic to S_3.

As before, we now prove a converse statement.

Proposition 5. *The surface X_3 is birational to:*

- *the double cover of a quadric Q branched along a line and a trigonal curve of genus six.*
- *the double cover of a Hirzebruch surface $\mathbb{F}_4$ branched along a curve with 4 nodes in $|3h|$ and the rational curve in $|s|$, where $h^2 = 4,\ s^2 = -4,\ (h, s) = 0$.*

Proof. By [N2, Theorem 4.2.2] the set of fixed points of σ_3 on X_3 is the disjoint union of a smooth curve C of genus 6 and a smooth rational curve L. Since $S_3 \subset \text{Pic}(X_3)$, X_3 admits an elliptic fibration π with a section and four singular fibers of Kodaira type I_2 or type III. Since any fiber of π is preserved by σ_3, then L is a section of π and C intersects each fiber in 3 points. Hence C has a triple cover to $\mathbb{P}^1$ and its ramification points are the singular points of irreducible fibers of π.

We will denote by $F_1, \ldots, F_4$ the singular fibers of π of type I_2 or III, by E_i the component of F_i meeting L and by E_i' the other component. Let $p : X_3 \to Y_3$ be the quotient by the involution σ_3. Note that $p(E_i)$ and $p(E_i')$ are (-1)-curves.

By contracting the curves $p(E_i)$, we get a smooth quadric surface. This gives the first assertion.

On the other hand, contracting the curves $p(E_i')$, we get a Hirzebruch surface $\mathbb{F}_4$ (note that the image of L has self-intersection -4). Since C intersects the ruling in 3 points, each E_i' at two points and it does not intersect L, then its image in $\mathbb{F}_4$ has 4 nodes and belongs to the class $3h$. This gives the second assertion. □

Corollary 4.

- *The divisor $\mathcal{H}_3$ is birational to the moduli space of pairs (C, L) where C is a trigonal curve of genus 6 and $L \in |K_C - 2g_3^1|$.*
- *The divisor $\mathcal{H}_{3a}^*$ parametrizes pairs (C, p) where C is trigonal and $p \in C$ or, equivalently, plane sextics with a node and a triple point.*
- *The divisor $\mathcal{H}_{3b}^*$ is birational to the moduli space of curves in $|3h|$ of $\mathbb{F}_4$ with 4 nodes.*

Proof. The first statement follows from Proposition 5 and the remarks at the beginning of this subsection since, by adjunction formula, the restriction of e to C coincides with $K_C - 2g_3^1$.

Given a trigonal curve $C \subset Q$ of genus six and $p \in C$, there exists a unique line $L \in |e|$ through p. This determines a K3 surface X with $S_3 \subset S_X$ as before. Moreover, the projection of C from p is a plane sextic with a triple point and a double point. The hyperplane class of $\mathbb{P}^2$ induces the linear system $K_C - g_3^1 - p$ on C and its pull-back to X is a nef class h with $h^2 = 2$.

Conversely, a generic point in $\mathcal{H}_{3a}^* \cup \mathcal{H}_{3b}^*$ gives a K3 surface X with $S_X \cong S_3 = U \oplus A_1^4$ and a degree two polarization h. Let e, f be a basis of U and $e_1, \ldots, e_4$ an orthogonal basis of A_1^4. Up to an isometry of S_3 we can assume that $r = e - f$ and that f gives an elliptic fibration on X. The orthogonal

complement $S_3 \cap r^\perp \cong S$ has two types of degree two polarizations: $h_j = 2(e+f) - \sum_{i=1}^4 e_i + e_j$ or $h = e + f$.

A point in $\mathcal{H}_{3b}^*$ gives a polarization h_b such that $h_b/2 = r/2$ in $A_T \cong A_S$, hence $h_b = h$. The class h_b contains r in the base locus and $2h_b$ maps X onto a cone over a rational normal quartic. In fact, the morphism associated to $2h_b$ is exactly the contraction of the curves $p(E_i')$ and the image of the curve L described in the proof of Proposition 5.

A point in $\mathcal{H}_{3a}^*$ gives a polarization $h_a = h_j$ for some $j = 1, \dots, 4$. In this case h_a has no base locus and gives a generically 2:1 map $X \to \mathbb{P}^2$. The branch locus of this map is a plane sextic with a triple point (in the image of r) and a node (in the image of e_j). The line through the two singular points intersects the sextic in one more point p. Hence this gives a pair (C, p), where C is trigonal and $p \in C$. □

Remark 3. The two irreducible components in $\mathcal{M}^*$ over $\mathcal{H}_3$ correspond to the components in $\mathcal{W}_6^2$ over the trigonal divisor in $\mathcal{M}_6$. With the notation in the proof of Proposition 2: the divisor $\mathcal{H}_{3a}^*$ corresponds to pairs $(C, D(p))$ and $\mathcal{H}_{3b}^*$ to $(C, 2g_3^1)$. This agrees with [Sh], where it is proved that the triple conic, which is the plane model of C associated to $2g_3^1$ (Proposition 2), "represents" K3 surfaces with a degree two polarization with a fixed component.

Remark 4. Let C be a plane sextic with four nodes $p_1, \dots, p_4$ such that p_1, p_2, p_3 lie on a line L. The blowing up of the plane in these points is a nodal del Pezzo surface Y (see section 1) and the double cover of Y branched along the proper transform of C is a K3 surface X. The pencil of lines through p_4 induces an elliptic fibration on X with general fiber f, 3 fibers of type I_2 and two sections s_1, s_2, given by the two (disjoint) inverse images of the line L. In particular, the Picard lattice of X contains the sublattice $S' = U \oplus A_1^3 \oplus < -4 >$, where U is generated by the fiber f and s_1, A_1^3 by the reducible components in each fiber and $< -4 >$ by $2f + s_1 - s_2$.

Conversely, let $r \in T$ be a primitive vector with $r^2 = -4$ such that $r/2 \in A_T$, then its orthogonal complement in T is isomorphic to $T' = U \oplus U \oplus E_8 \oplus A_1^3 \oplus < -4 >$ and $T'^\perp \cong S'$.

By choosing a different blow-down map for Y we get a plane sextic with a tacnode and two nodes. In fact, the elliptic fibration described above is induced by the pencil of lines through the tacnode.

4. Compactifications

4.1. **Satake-Baily-Borel compactification.** The moduli spaces $\mathcal{M}$, $\mathcal{M}^*$ are quasi-projective algebraic varieties. Since they are arithmetic quotients of a symmetric bounded domain, we can consider their Satake-Baily-Borel (SBB) compactifications $\overline{\mathcal{M}}$ and $\overline{\mathcal{M}^*}$ (see [BB] and [Sc],§ 2).

It is known that boundary components of the SBB compactification are in bijection with primitive isotropic sublattices of T up to Γ and Γ^* respectively, such that k-dimensional boundary components correspond to rank $k+1$ isotropic sublattices. Since T has signature $(2, 15)$, the boundary components will be either 0 or 1 dimensional.

Lemma 9. *Let $\mathcal{I}$ be the set of primitive isotropic vectors in T. There are two orbits in $\mathcal{I}$ with respect to the action of* Γ :

$$\mathcal{I}_1 = \{v \in \mathcal{I} : (v, T) = \mathbb{Z}\} \qquad \mathcal{I}_2 = \{v \in \mathcal{I} : (v, T) = 2\mathbb{Z}\}.$$

There are three orbits with respect to Γ^* : $\mathcal{I}_1$ *and two orbits decomposing* $\mathcal{I}_2$.

Proof. By Proposition 4.1.3 in [Sc] there is a bijection between orbits of isotropic vectors in T modulo Γ (Γ^*) and isotropic vectors in A_T modulo the induced action of Γ (Γ^*). By Lemma 4 the map

$\Gamma \to \mathrm{O}(q_T)$ is surjective and clearly the image of Γ^* is given by elements of $\mathrm{O}(q_T)$ fixing ξ_1. Then it follows from Lemma 3 that there are exactly two orbits of isotropic vectors in A_T for the action of Γ and three for the action induced by Γ^*. $\square$

Corollary 5. *The boundaries of $\overline{\mathcal{M}}$ and $\overline{\mathcal{M}}^*$ contain two and three zero-dimensional components respectively.*

We will denote by p, q the zero-dimensional boundary components of $\overline{\mathcal{M}}$ corresponding to the orbits $\mathcal{I}_1, \mathcal{I}_2$ in Lemma 9 respectively and with q_1, q_2 the zero-dimensional boundary components of $\overline{\mathcal{M}}^*$ corresponding to the orbits of Γ^* decomposing $\mathcal{I}_2$.

Remark 5. By [N1, Theorem 3.6.2] there is also an isomorphism

$$T \cong U \oplus U(2) \oplus A_1 \oplus D_4 \oplus E_8.$$

In the following we will denote by e, f and e', f' the standard bases of U and $U(2)$, by β a generator of A_1, by $\gamma_1, \ldots, \gamma_4$ and $\alpha_1, \ldots, \alpha_8$ the standard root bases of D_4 and E_8. Note that $e, f \in \mathcal{I}_1$ and $e', f' \in \mathcal{I}_2$.

We now classify one dimensional boundary components in $\overline{\mathcal{M}}$ by studying Γ-orbits of primitive isotropic planes in T. We will say that such a plane is of *type* (i, j), $i, j = 1, 2$ if it is generated by a vector in $\mathcal{I}_i$ and one in $\mathcal{I}_j$.

Let $\mathcal{G}_1$ be the genus of $E_8 \oplus A_1^5$ and let $\mathcal{G}_2$ be the genus of $E_8 \oplus A_1 \oplus D_4$. If N is a lattice in $\mathcal{G}_1$, then $T \cong U \oplus U \oplus N$ by [N1, Theorem 3.6.2]. By taking two isotropic vectors, each in one copy of U, we get an isotropic plane in T of type $(1, 1)$. Similarly, if $N_2 \in \mathcal{G}_2$ then $T \cong U \oplus U(2) \oplus N_2$ and the plane generated by a generator of U and one of $U(2)$ is isotropic of type $(1, 2)$.

Lemma 10. *The isomorphism classes of lattices in $\mathcal{G}_1$ and $\mathcal{G}_2$ are given in the following table.*

	R	$\mathcal{G}_1$	$\mathcal{G}_2$
a	E_8^3	$E_7 \oplus D_4 \oplus A_1^2$, $D_6^2 \oplus A_1$, $E_8 \oplus A_1^5$	$A_1 \oplus E_8 \oplus D_4$
b	$E_7^2 \oplus D_{10}$	$\overline{E_7 \oplus A_1^6}, \overline{D_6 \oplus D_4 \oplus A_1^3}, D_8 \oplus D_4 \oplus A_1, \overline{D_8 \oplus A_1^5}, D_{10} \oplus A_1^3$	$E_7 \oplus D_6, \overline{D_{10} \oplus A_1^3}$
c	$D_{16} \oplus E_8$	$\overline{D_8 \oplus A_1^5}$	$D_{12} \oplus A_1$
d	$A_{17} \oplus E_7$	$(A_1^4)^\perp$ in A_{17}	

TABLE 1. One dimensional boundary components

Proof. The orthogonal complements of $E_8 \oplus A_1^5$ and $E_8 \oplus A_1 \oplus D_4$ in E_8^3 are isomorphic to $R_1 = E_7 \oplus A_1^4$ and $R_2 = E_7 \oplus D_4$ respectively. By Proposition 6.1.1, [Sc] the isomorphism classes in $\mathcal{G}_1$ and $\mathcal{G}_2$ can be obtained by taking the orthogonal complements of primitive embeddings of R_1 and respectively R_2 into even negative definite unimodular lattices of rank 24, i.e. Niemeier lattices. These lattices are uniquely determined by their root sublattice R, hence they are denoted by $N(R)$ (see [CS], Chap. 18). In order to determine all lattices in the $\mathcal{G}_i$ we first classify all primitive embeddings of R_1, R_2 into R and take their orthogonal complements $R_i^\perp$ in R. Then we take the primitive overlattice

$\overline{R_i^\perp}$ of $R_i^\perp$ in $N(R)$ which contains $R_i^\perp$ as a subgroup of index at most 2. Here we have used the classification of embeddings between root lattices due to Nishiyama [Ni]. This gives isomorphism classes $\overline{R_i^\perp}$ in $\mathcal{G}_i$. In Table 1 all root lattices R appear such that R_i can be embedded in $N(R)$ and the corresponding lattices in $\mathcal{G}_1$ and $\mathcal{G}_2$. If $R_i^\perp$ is primitive in $N(R)$, then we omit the overline. □

Theorem 2. *The boundary of* $\overline{\mathcal{M}}$ *contains* 14 *one dimensional components* $B_1, \dots, B_{14}$ *where the closure of* B_i, $i = 1, \dots, 10$ *contains only* p *and the closure of* B_j, $j = 11, \dots, 14$ *contains both* p *and* q.

Proof. As remarked before, to the lattices in $\mathcal{G}_1$ we can associate isotropic planes of type $(1,1)$ in T which are not Γ-equivalent. Conversely, by Lemma 5.2 in [Sc], any isotropic plane E of type $(1,1)$ can be embedded in $U \oplus U$ and $T \cong U \oplus U \oplus E^\perp/E$ where $E^\perp/E \in \mathcal{G}_1$. Hence, boundary components containing only p are in one-to-one correspondence with lattices in $\mathcal{G}_1$.

The proof is more subtle for isotropic planes of type $(1,2)$. Note that if $v \in T$ is a primitive isotropic vector of type 2 and E is an isotropic plane containing v, then E determines a primitive vector in $M_v = v^\perp/\mathbb{Z}v$. Hence, isotropic planes of type $(1,2)$ correspond to orbits of isotropic vectors in M_v. In this case $M_v \cong U \oplus E_8 \oplus D_4 \oplus A_1$ and orbits of isotropic vectors can be determined by Vinberg's algorithm (see §1.4 [V] or §4.3 [St]).

By [N2, Theorem 0.2.3], the Weyl group $W(M_v)$ has finite index in $O(M_v)$. This implies that the algorithm will finish in a finite number of steps. To start the algorithm we fix the vector $\bar{x} = e + f$. Then at each step we have to choose roots $x \in M_v$ such that the *height*

$$h = \frac{(x, \bar{x})}{\sqrt{-x^2}}$$

is minimal and $(x_i, x_j) \geq 0$ for $j = 1, \dots, i-1$. In our case we get:

$i)$ $(x, \bar{x}) = 0$: $\quad u := e - f,\ \alpha_1, \dots, \alpha_8, \gamma_1, \dots, \gamma_4, \beta.$
$ii)$ $(x, \bar{x}) = 1$: $\quad \alpha := f + \bar{\alpha}_8,\ \gamma := f + \bar{\gamma}_1,\ \beta' := f - \beta.$
$iii)$ $(x, \bar{x}) = 4$: $\quad \delta_i := 2(e+f) - \beta + \bar{\alpha}_1 + \bar{\gamma}_i,\ \ j = 2, 3, 4.$
$iv)$ $(x, \bar{x}) = 12$: $\quad \alpha' := 6(e+f) - 3\beta + 2\bar{\alpha}_4 + \bar{\gamma}_2 + \bar{\gamma}_3 + \bar{\gamma}_4$

where $\bar{\alpha}_1, \dots, \bar{\alpha}_8$ and $\bar{\gamma}_1, \dots, \bar{\gamma}_4$ are the dual bases of E_8 and D_4. We now draw the Dyinkin diagram associated to these roots. Let $g_{ij} = (e_i, e_j)/\sqrt{e_i^2 e_j^2}$. Then two vertices i, j corresponding to vectors e_i, e_j are connected by

$$\begin{array}{lcl}
\bullet \qquad \bullet & \text{if} & g_{ij} = 0, \\
\bullet \text{———} \bullet & \text{if} & g_{ij} = 1/2, \\
\bullet - - - \bullet & \text{if} & g_{ij} = 1, \\
\bullet \text{—}\Vert\text{—} \bullet & \text{if} & g_{ij} > 1.
\end{array}$$

The diagram in our case is given in Figure 1 (see also Figure 5, [Ko]). Note that the symmetry group of the diagram is $\mathbb{Z}_2 \times S_3$ and it can be easily seen that all symmetries can be realized by isometries in Γ. The maximal parabolic subdiagrams of rank 13 are of four types :

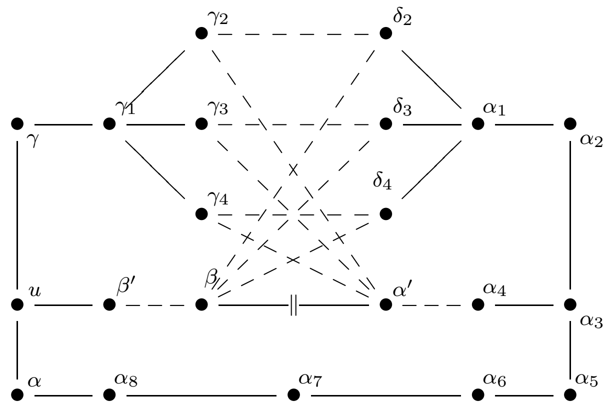

FIGURE 1. The Dynkin diagram of $W(M_v)$

$$\tilde{E}_8 \oplus \tilde{D}_4 \oplus \tilde{A}_1 = \langle \alpha_i, \alpha, \beta', \beta, \gamma, \gamma_j \rangle \quad i = 1, \ldots, 8;\ j = 1, \ldots, 4$$

$$\tilde{D}_{12} \oplus \tilde{A}_1 = \langle \alpha_i, \alpha, u, \gamma, \gamma_j, \beta, \delta_4 \rangle \quad i = 2, \ldots, 8;\ j = 2, 3$$

$$\tilde{E}_7 \oplus \tilde{D}_6 = \langle \alpha_i, \delta_2, \alpha, u, \beta', \gamma, \gamma_j \rangle \quad i = 1, \ldots, 7;\ j = 3, 4.$$

$$\tilde{D}_{10} \oplus \tilde{A}_1^3 = \langle \alpha_i, \alpha, u, \gamma, \beta', \gamma_j, \delta_j \rangle \quad i = 2, \ldots, 8;\ j = 2, 3, 4.$$

Note that each type is an orbit for the action of Γ. These subdiagrams correspond to non-equivalent isotropic vectors in M_v. Hence, we get 4 isotropic planes in T containing a vector in $\mathcal{I}_2$ and a direct analysis shows that all of them are of type $(1, 2)$. □

It follows from the proof of Theorem 2 that the boundary components of $\overline{\mathcal{M}}$ are in one-to-one correspondence with the lattices in $\mathcal{G}_1$ and $\mathcal{G}_2$. These lattices appear in connection to degenerations of K3 surfaces as explained for example in [Sc]. This allows to compare the SBB compactification with more geometrically meaningful compactifications, as the ones obtained by means of geometric invariant theory.

In case of K3 surfaces with a degree two polarization this is well-understood ([Sh], [F], [Lo2]). Table 2 describes the correspondence between type II boundary components of the GIT compactification of plane sextics and one dimensional boundary components of the Baily-Borel compactification for degree two K3 surfaces. The lattice appearing in the SBB column is $E^{\perp}/E$, where E is the isotropic lattice associated to the boundary component.

Remark 6. In the proof of Theorem 2 we showed that boundary components of $\overline{\mathcal{M}}$ containing only p in their closure correspond to primitive embeddings of the lattice $E_7 \oplus A_1^4$ into Neimeier lattices. Equivalently, they correspond to primitive embeddings of the lattice A_1^4 in the root lattices $E_8 \oplus E_8$, $E_7 \oplus D_{10}$, D_{16}, A_{17}. Note that a double cover branched over a node has an A_1 singularity hence, embedding A_1^4 in the root lattices is equivalent to choose a distribution of the 4 nodes on the corresponding configurations in Table 2 (where more than one node can "collapse" to the same singular point of the configuration).

	GIT	SBB
IIa:	$(x_0x_2 + a_1x_1^2)(x_0x_2 + a_2x_1^2)(x_0x_2 + a_3x_1^2) = 0$	$E_8 \oplus E_8 \oplus A_1$
IIb:	$x_2^2 f_4(x_0, x_1) = 0.$	$E_7 \oplus D_{10}$
IIc:	$(x_0x_2 + x_1^2)^2 f_2(x_0, x_1, x_2) = 0.$	$D_{16} \oplus A_1$
IId:	$f_3(x_0, x_1, x_2)^2 = 0.$	A_{17}

TABLE 2. GIT and SBB of plane sextics

For example, let q_1, q_2 be the two singular points in the IIa configuration. We can either embed one node in q_1 and 3 nodes in q_2 (this gives the root lattice $E_7 \oplus D_1 \oplus A_1$), two nodes in q_1 and two in q_2 (this gives the root lattice $D_6^2 \oplus A_1$) or 4 nodes in q_1 (this gives the root lattice $E_8 \oplus A_1^5$).

Similarly, boundary components containing both p and q in their closure correspond to embeddings of the lattice D_4 into the previous root lattices. Note that a double cover branched over a triple point has a D_4 singularity.

In fact we conjecture that a one dimensional boundary component B of $\overline{\mathcal{M}}$ of type a, b, c or d (see Table 1) corresponds to a boundary component of type IIa, IIb, IIc or IId respectively with

- 4 marked nodes (eventually collapsing) if $q \notin B$
- a marked triple point if $q \in B$.

Note that the configuration IId has no triple points, in fact there is no one-dimensional boundary component of type d containing q in its closure.

Remark 7. By corollaries 3 and 4 the moduli space $\mathcal{M}$ contains two divisors which are birational to $\mathbb{P}^2$ and $\mathbb{P}^1$ fibrations over the locus of plane quintics and trigonal curves respectively. This suggests that we need to blow-up the moduli space of curves of genus six in order to extend the period map to these loci.

Bi-elliptic and hyperelliptic curves of genus six are mapped to one dimensional boundary components of $\overline{\mathcal{M}}$. In fact, the configuration IIc is a plane model for hyperelliptic curves and case IId is the plane model of a bi-elliptic curve of genus six (see §1).

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